Mechanics and Relativity for Mathematicians

- Tue Apr 1 2025: 15:00 - 17:00 -

Write your name and student number on **all** sheets. There are four problems in this exam. You can earn 90 points in total. You are allowed to bring one (hand-written) two-sided A4 page as 'cheat sheet'.

Question 1: Planetary Motion (8+8+8)

A comet is moving under Earth's gravitational influence along a parabolic trajectory. Assume that Earth is much more massive than the comet and neglect any external forces apart from gravity.

(a) What quantities are conserved for the comet? Under what conditions are these quantities conserved? Express these conditions in terms of the force $\vec{F}$ and/or the potential energy V.

(b) What is the total energy of the comet? At which point along its trajectory does the comet attain its highest speed?

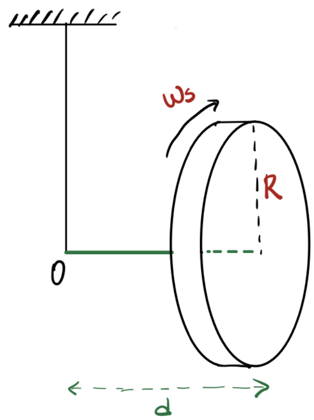
(c) Suppose that while the comet is approaching, Earth is suddenly replaced by another planet with a stronger gravitational field (i.e., greater mass). How would this affect the comet's trajectory? Would the new orbit still be parabolic? Explain your reasoning briefly.

(a) Total mechanical energy: conserved in the absence of non-conservative forces – there exists gravitational potential: $\vec{F}_g = -\vec{\nabla}V$. Angular momentum: conserved because $\vec{F}_g = F_g\hat{r}$. 2 pts for each correctly identified conserved quantity, and 2 pts for each correct condition.

(b) $E = 0$. The comet reaches its highest speed at the point where it is closest to Earth. 4 points for correctly determining the energy, and 4 points for identifying the location where the comet attains its highest speed.

(c) The comet's orbit would become elliptical instead of parabolic due to the stronger gravitational force of the new planet, which lowers the gravitational potential and makes the comet's total energy negative. In planetary motion, a negative total energy results in an elliptical orbit. 4 points for correctly determining the new trajectory, and 4 points for correct explanation

Question 2: Rigid Body Motion (4+8+4+8)
A uniform disk, with radius R and mass m, is suspended from the end of its axle, which has length d, by a rope and spun up, as shown below. It is then placed horizontally and as the disk spins, it begins to rotate around the center point O. The spin angular speed ω_s is much greater than the precession angular speed ω_p. Assume that the disk remains vertical and does not fall during its precession.



(a) What is the direction of the spin angular momentum $\vec{L}$ of the disk at the given instant, as shown in the drawing?

(b) What causes precession? Find the direction of precession. Explain your answer.

(c) Derive the moment of inertia of a uniform disk about its center of mass.

(d) Find the precession angular velocity ω_p in terms of m, R, ω_s, d and the gravitational acceleration g (Use $I_{\mathrm{CM}} = amR^2$ as the moment of inertia of a disk about its center of mass, where a is a constant, if you did not determine it in the previous question.).

(a) The angular momentum points center O as shown below.

(b) The force of gravity/ the torque due to gravity causes precession. The direction of $\Delta\vec{L}$ has to be the same as the direction of torque, thus it is in the clockwise direction. 2 pts for the correct cause, 2 pts for the direction of precession, 4 pts for the correct explanation

(c) The moment of inertia of a uniform disk about its center of mass is given by:

$$I_{CM} = \int r^2\, dm$$

where r is the distance from the axis of rotation and dm is the mass element.

Expressing dm in terms of the surface mass density:

$$\sigma = \frac{m}{\pi R^2}, \quad dm = \sigma\, dA = \frac{m}{\pi R^2} \cdot 2\pi r\, dr$$

Substituting these into the integral and evaluating the integral correctly:

$$I_{CM} = \frac{2m}{R^2} \int_0^R r^3 \, dr = \frac{2m}{R^2} \cdot \frac{R^4}{4} = \frac{mR^2}{2}$$

Thus, the moment of inertia of a uniform disk about its center of mass is:

$$I_{CM} = \frac{1}{2} mR^2$$

2 points for correctly formulating the integral and 2 points for correct value of I.

(d) When the disk precesses by a small angle $d\theta$, the angular momentum will rotate as well. The angle between the new and old angular momenta will be $d\theta$. The new angular momentum, the old angular momentum, and the small change in angular momentum form a triangle. Considering only the magnitudes of these vectors, we can write:

$$dL_s = L_s \sin(d\theta) \approx L_s d\theta = I_{CM} \omega_s d\theta.$$

The torque τ is related to the rate of change of angular momentum:

$$\tau = \frac{dL_s}{dt} = I_{CM} \omega_s \omega_p.$$

Since the torque is also given by the gravitational force $\tau = Mgd$, we have:

$$I_{CM} \omega_s \omega_p = Mgd \quad \Rightarrow \quad \omega_p = \frac{2gd}{R^2 \omega_s}.$$

2 points for correctly determining the torque due to gravity, 4 points for correctly computing the change in angular momentum, and 2 points for the correct expression for ω_p.

Question 3: Turntable (8+8)

A turntable rotates counterclockwise with constant angular velocity $\vec{\omega}$ (as viewed from above). A person stands at a radial distance r_0 from the center, facing tangentially in the direction of motion. Holding the ball such that the ball, the person, and the center of the turntable are aligned, they drop it from a height. The ball is initially at rest relative to the person, and gravity g acts downward.

(a) Determine the magnitudes and directions of the fictitious forces acting on the ball in the turntable frame at the moment the ball is released ($t = 0$).

(b) Does the ball land in front of or behind the person on the turntable as it falls? Explain your reasoning, considering the effects of the Coriolis force and centrifugal force in the rotating frame.

(a) At $t = 0$: The ball is at rest in the rotating frame, so no Coriolis force acts on it. The only fictitious force is the centrifugal force acting radially outward with magnitude:

$$\mathbf{F}_{\text{cent}} = m\omega^2 r_0\,\hat{r}$$

4 points for the absence of Coriolis force, 2 points for correctly identifying the magnitude of the centrifugal force, and 2 points for correctly identifying its direction.

(b) At later times: As the ball falls, the centrifugal force gives it a small outward radial velocity v_r. This velocity induces a Coriolis force acting opposite to the direction of rotation (clockwise):

$$\mathbf{F}_C = -2m\boldsymbol{\omega} \times \mathbf{v}_r$$

The ball lands behind the person in the rotating frame, as the Coriolis force causes a drift opposite to the direction of rotation. 4 points for the correct answer, 4 points for the correct explanation.

Question 4: Train in a tunnel (8+8+10)

A train and a tunnel both have proper lengths L. The train moves toward the tunnel at speed $0.8c$. A bomb is located at the front of the train. The bomb is designed to explode when the front of the train passes the far end of the tunnel. A deactivation sensor is located at the back of the train. When the back of the train passes the near end of the tunnel, the sensor tells the bomb to disarm itself. Does the bomb explode? To answer this, follow the following steps:

(a) In the frame of the tunnel, what is the length of the tunnel and train? What fraction of the train is inside the tunnel when the front of the train passes the far end of the tunnel?

(b) In the frame of the train, what is the length of the tunnel and train? What fraction of the train is inside the tunnel when the far end of the tunnel passes the front of the train?

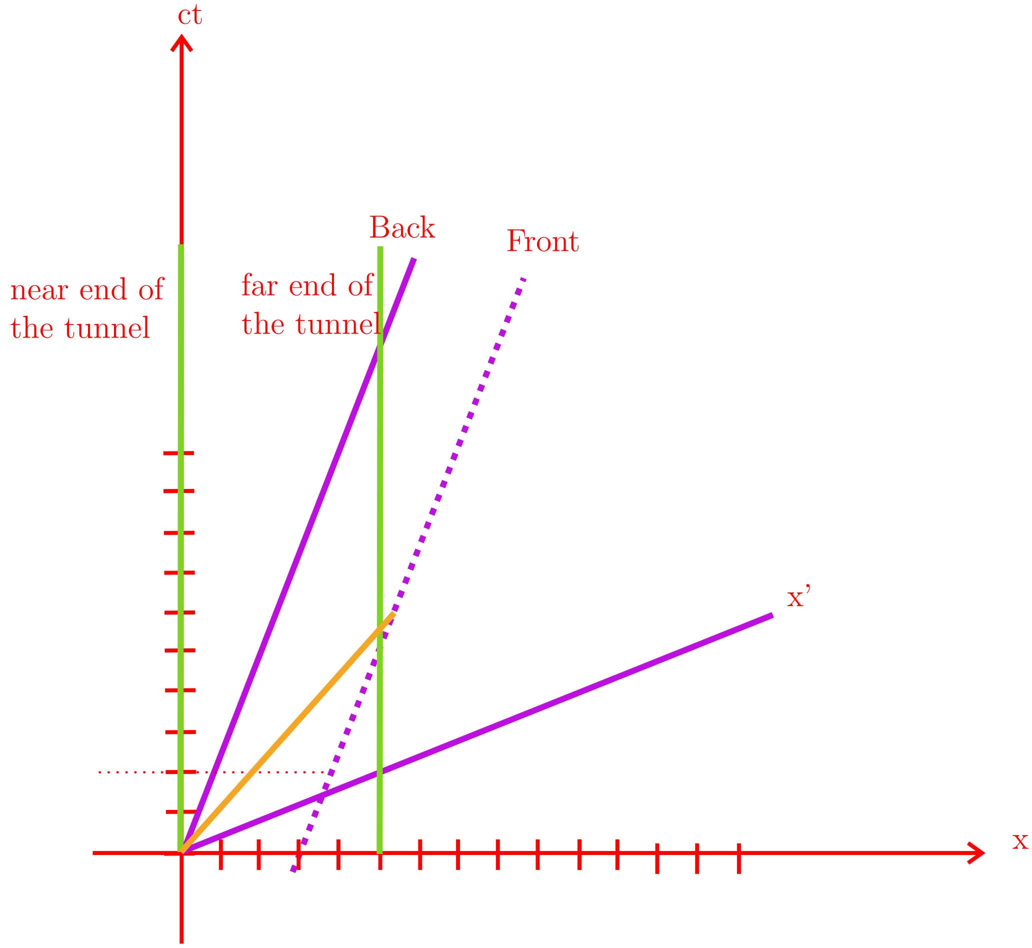
(c) Draw a two-observer diagram on the graph paper provided, from the perspective of the tunnel frame. Include the relevant worldlines – the far and near ends of the tunnel, as well as the front and back of the train. Indicate the event when the deactivation sensor is triggered and show whether the sensor can disarm the bomb. Clearly state whether the bomb explodes in both reference frames and briefly explain your reasoning.

(a) In the frame of the tunnel, the train shrinks to $\frac{L}{\gamma}$ so the moment the front end reaches the far end of the tunnel, the whole train is inside 4 points for correctly determining the length of the train in the tunnel frame(2 points out of 4 awarded for the correct Lorentz factor), 2 points for identifying the length of the train in the train's rest frame, and 2 points for stating the fraction of the train that is inside the tunnel.

(b) In the frame of the train, the tunnel shrinks to $\frac{L}{\gamma}$ so the moment the front end reaches the far end of the tunnel, $\frac{\gamma-1}{\gamma}L$ part of the train lays outside. 4 points for correctly determining the length of the tunnel in the tunnel frame, 2 points for identifying the length of the train in the train's rest frame, and 2 points for stating the fraction of the train that is inside the tunnel.

(c) Yes, the bomb explodes. This is clear in the frame of the train. However, in the tunnel frame, the deactivation device gets triggered before the front of the train passes the far end of the tunnel, so you might think that the bomb does not explode. It seems we have a paradox.

The resolution to this paradox is that the deactivation device cannot instantaneously tell the bomb to deactivate itself. It takes a finite time for the signal to travel the length of the train from the sensor to the bomb. And it turns out that this transmission time makes it impossible for the deactivation signal to get to the bomb before the bomb gets to the far end of the tunnel, no matter how fast the train is moving.



The correct spacetime diagram is worth 6 points: 1 points for each worldline (totaling 4 points) with correct slope, 1 points for ensuring the correct length ratio between the tunnel and the train, and 1 points for properly illustrating the signal traveling from the back to the front of the train. 2 points are awarded for the correct answer and 2 points for the explanation.